CHAPTER 19

Circle theorems

In this chapter you will learn how to:
• use correct vocabulary associated with circles
• use tangent properties to solve problems
• prove and use various theorems about angle properties inside a circle
• prove and use the alternate segment (intersecting tangent and chord) theorem.

You will also be challenged to:
• investigate the nine-point circle theorem.

Starter: Circle vocabulary

Here are some words you will often encounter when working with circles:
Centre Radius Chord Diameter Circumference Tangent Arc Sector Segment

Try to match the correct words to the nine diagrams below:

10 a) The French call a rainbow an ‘arc-en-ciel’ (ciel = sky).
   Do you think this is a good name?
b) The English call a piece of an orange a ‘segment’.
   Do you think this is a good name?
19.1 Tangents, chords and circles

In this section you will learn some theorems about circles, and then use them to solve problems. The theorems are concerned with tangents and chords.

A **chord** is a line segment joining two points on the circumference of a circle.

A **tangent** is a straight line that touches a circle only once.

A line segment drawn from the centre of a circle to the midpoint of a chord will intersect the chord at right angles.

A tangent and radius meet at right angles.

The fact that the radius and the tangent meet at right angles is very obvious when one is vertical and the other is horizontal... but is not quite so obvious when the situation is rotated, like this.
The two external tangents to a circle are equal in length.

These theorems may be used to help you determine the values of missing angles in circles. When you use them, remember to tell the examiner which theorem(s) you have used.

**EXAMPLE**

The diagram shows a circle, centre O. PT is a tangent to the circle. Find the value of $x$.

**SOLUTION**

Angle OPT $= 90^\circ$ (angle between the radius and tangent is $90^\circ$).

The angles in triangle TOP add up to $180^\circ$, so:

\[
x + 48^\circ + 90^\circ = 180^\circ \\
x + 138^\circ = 180^\circ \\
x = 180^\circ - 138^\circ \\
x = 42^\circ
\]
EXAMPLE

The diagram shows a circle, centre Q.
AB is a chord across the circle.
M is the midpoint of AB.
Find the value of \( y \).

SOLUTION

Angle AMQ = 90° (radius bisecting chord).
The angles in triangle AMQ add up to 180°.
So:
\[
\begin{align*}
  y & + 62° + 90° = 180° \\
  y & + 152° = 180° \\
  y & = 180° - 152° \\
  y & = 28°
\end{align*}
\]

Since a radius bisects a chord at right angles, there is often an opportunity to use Pythagoras’ theorem.

EXAMPLE

The diagram shows a radius OT that bisects the chord AB at M. MB = 12 cm.
The radius of the circle is 13 cm.
Work out the length MT.

SOLUTION

First join OB:

Now apply Pythagoras’ theorem to triangle OBM:
\[
OM^2 = 13^2 - 12^2 \\
OM = \sqrt{169 - 144} \\
OM = \sqrt{25} = 5\text{ cm}
\]
The distance OT is a radius, that is, 13 cm.
Thus MT = 13 - 5 = 8 cm
EXERCISE 19.1

1 PT is a tangent to the circle, centre O. Angle PTO = 29°.

![Diagram of circle with tangent PT and angle PTO = 29°]

a) State, with a reason, the value of the angle marked \( x \).
b) Work out the value of the angle marked \( y \).

2 PT is a tangent to the circle, centre O. PT = 24 cm, OP = 7 cm.

![Diagram of circle with tangent PT and radius OP]

The line OT intersects the circle at R, as shown. Work out the length of RT.

3 TP and TR are tangents to the circle, centre O. Angle PTR is 44°.

![Diagram of circle with tangents TP and TR and angle PTR = 44°]

a) Work out the size of angle POR. Give reasons.
b) What type of quadrilateral is OPTR? Explain your reasoning.
4 TP and TR are tangents to the circle, centre O. Angle POR is 130°.

a) What type of triangle is triangle OPR?
b) Work out the value of \( x \).
c) Work out the value of \( y \).

5 The diagram shows a circle, centre O. The radius of the circle is 5 cm. M is the midpoint of EF. OM = 3 cm.

Calculate the length of EF.

6 The diagram shows a circle, centre O. AB = 34 cm. M is the midpoint of AB. OM = 8 cm.

Work out the radius of the circle.
7 From a point T, two tangents TP and TQ are drawn to a circle, centre O.
   a) Make a sketch to show this information.
   b) The length TQ is measured, and found to be exactly the same as the length PO.
      What type of quadrilateral is TPOQ?

8 The diagram shows a circle, centre O.
   AB and CD are chords.
   The radius OT passes through the midpoints M and N of the chords.
   OM = 8 cm, NT = 3 cm, AB = 30 cm.

   a) Explain why angle AMO = 90\degree.
   b) Use Pythagoras’ theorem to calculate the distance AO.
      Show your working.
   c) Calculate the distance MN.
   d) Calculate the length of the chord CD.
19.2 Angle properties inside a circle

There are several important theorems about angles inside a circle. You will need to learn these, and use them to solve numerical problems. You may also be asked to prove why they are true.

Consider two points, A and B say, on the circumference of a circle. The angle subtended by the arc AB at the centre is angle AOB.

There is a theorem in circle geometry which states that angle AOB is exactly twice angle AXB.

This result is quite easy to prove, and is the basic theorem upon which several other circle theorems are built.
THEOREM

The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference of the circle.

PROOF

Make a diagram to show the arc AB, the centre O, and the point X on the circumference of the circle:

From X, draw a radius to O, and produce it, which means extend it slightly:

Triangle AOX is isosceles, since both OA and OX are radii of the same circle. Therefore angles OAX and OXA are equal. These are marked on the diagram with a letter a:

Likewise the triangle BOX is isosceles, since both OB and OX are radii of the same circle. Therefore angles OBX and OXB are equal. These are marked on the diagram with a letter b:

The angle at the circumference is angle AXB = a + b.
To obtain an expression for the angle at the centre, look at this magnified copy of the diagram:

![Diagram showing angles AOX, AYX, BOX, BZY, and OZX.]

Angle AOX = 180° - a - a = 180° - 2a

So: angle AOX = 180° - (180° - 2a) = 2a

Likewise:

angle BOX = 180° - b - b = 180° - 2b

So: angle BOY = 180° - (180° - 2b) = 2b

Thus the angle at the centre is:

Angle AOB = 2a + 2b
            = 2(a + b)

But angle AXB = a + b, from above.

Therefore angle AOB = 2 \times angle AXB.

Thus, the angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference of the circle.

Two further theorems can be deduced immediately from this first one. You can quote the previous theorem to justify these proofs.
**Theorem**

Angles subtended by an arc in the same segment of a circle are equal.

![Diagram showing angles subtended by an arc in the same segment of a circle.]

**Proof**

Join AO and OB so that they form an angle at the centre:

If the angle at P is \( x \), then the angle at the centre must be \( 2x \).

...and if the angle at the centre is \( 2x \), then the angle at Q must be \( x \).

Thus angles APB and AQ are equal.

**Theorem**

The angle subtended in a semicircle is a right angle.

**Proof**

Since AB is a diameter, AOB is a straight line.

Thus angle AOB = \( 180^\circ \).

Using the result that the angle at the circumference is half that at the centre:

\[
\text{Angle APB} = \frac{180^\circ}{2} = 90^\circ
\]

If this line is a diameter...

...then this angle will be a right angle.
**EXAMPLE**

Find the values of the angles marked $x$ and $y$. Explain your reasoning in both cases.

**SOLUTION**

Angle $x = 44^\circ$ (angles in the same segment are equal).

$\angle x + 70^\circ + y = 180^\circ$ (angles in a triangle add up to $180^\circ$) and $x = 44^\circ$, so:

\[
44^\circ + 70^\circ + y = 180^\circ \\
114^\circ + y = 180^\circ \\
y = 180^\circ - 114^\circ \\
y = 66^\circ
\]

**EXAMPLE**

Find the values of the angles marked $x$ and $y$. Explain your reasoning in both cases.

**SOLUTION**

$x = 70 - 2 = 58^\circ$ (angle at centre $= 2 \times$ angle at circumference)

$y = 180 - 90 - 62 = 28^\circ$ (angle in a semicircle is a right angle and angles in a triangle add up to $180^\circ$)
EXERCISE 19.2

Find the missing angles in these diagrams, which are not drawn to scale. Explain your reasoning in each case.
19.3 Further circle theorems

**THEOREM**

The angles subtended in opposite segments add up to 180°.

**PROOF**

Denote the angles APB and AQB as \( p \) and \( q \) respectively.

Then the angles at the centre are twice these, that is, \( 2p \) and \( 2q \).

Angles at point O add up to 360°, so:

\[
2p + 2q = 360°
\]

Thus \( 2(p + q) = 360° \)

So \( p + q = 180° \)

The points A, P, B and Q form a quadrilateral whose vertices lie around a circle; it is known as a *cyclic quadrilateral*. Thus the theorem may also be stated as:

Opposite angles of a cyclic quadrilateral add up to 180°
EXAMPLE

Find the angles $x$ and $y$.

\[ \text{Diagram not to scale} \]

SOLUTION

For angle $x$, we have $x + 116^\circ = 180^\circ$ (angles in opposite segments)
Thus $x = 180^\circ - 116^\circ = 64^\circ$

For angle $y$, two construction lines are needed:

Use angles in opposite segments, $u = 180 - 96 = 84^\circ$.

Using the angle at centre is twice the angle at circumference:

\[ y = 84 \times 2 = 168^\circ \]
THEOREM

The angle between a tangent and chord is equal to the angle subtended in the opposite segment. (This is often called the alternate segment theorem.)

PROOF

First, consider the special case of a tangent meeting a diameter:

Since the angle in a semicircle is 90°, the other two angles in the triangle add up to 90°.
Hence \( y + z = 90° \).

Since a radius and tangent meet at 90°:
\[ x + z = 90° \]
Hence \( x + z = y + z \).
From which it follows that \( x = y \).

Now move \( P \) around the circle to \( Q \), say, so that it is no longer on the end of a diameter. The angle at \( Q \) is equal to the angle at \( P \), as they are angles in the same segment. Thus the theorem is proved.
EXERCISE 19.3

Find the missing angles in these diagrams, which are not drawn to scale. Explain your reasoning in each case.
REVIEW EXERCISE 19

1 A circle of diameter 10 cm has a chord drawn inside it. The chord is 7 cm long.
   a) Make a sketch to show this information.
   b) Calculate the distance from the midpoint of the chord to the centre of the circle.
      Give your answer correct to 3 significant figures.

2 The diagram shows a circle, centre O.
   PT and RT are tangents to the circle. Angle POR = 144°.
   a) Work out the size of angle PTR, marked x.
   b) Is it possible to draw a circle that passes through the four points P, O, R and T?
      Give reasons for your answer.

3 The diagram shows a circle, centre O.
   PT and RT are tangents to the circle. Angle PTR = 32°.
   a) Work out the size of angle PSR, marked y.
      Hint: Draw in OP and OR.
   b) Is it possible to draw a circle that passes through the four points P, S, R and T?
      Give reasons for your answer.
4 In the diagram, A, B and C are points on the circle, centre O.
Angle BCE = 63°. FE is a tangent to the circle at point C.

a) Calculate the size of angle ACB. Give reasons for your answer.

b) Calculate the size of angle BAC. Give reasons for your answer.

5 P, Q, R and S are points on the circumference of a circle, centre O.
PR is a diameter of the circle. Angle PSQ = 56°.

a) Find the size of angle PQR. Give a reason for your answer.

b) Find the size of angle PRO. Give a reason for your answer.

c) Find the size of angle POQ. Give a reason for your answer.
6. A, B, C and D are four points on the circumference of a circle. ABE and DCE are straight lines. Angle BAC = 25°. Angle EBC = 60°.
   a) Find the size of angle ADC.
   b) Find the size of angle ADB.

   Angle CAD = 65°. Ben says that BD is a diameter of the circle.
   c) Is Ben correct? You must explain your answer.

7. The diagram shows a circle, centre O. AC is a diameter. Angle BAC = 35°. D is the point on AC such that angle BDA is a right angle.
   a) Work out the size of angle BCA. Give reasons for your answer.
   b) Calculate the size of angle DBC.
   c) Calculate the size of angle BOA.
8. A, B, C and D are four points on the circumference of a circle. TA is the tangent to the circle at A. Angle DAT = 30°. Angle ADC = 132°.

a) Calculate the size of angle ABC. Explain your method.

b) Calculate the size of angle CBD. Explain your method.

c) Explain why AC cannot be a diameter of the circle.

9. Points A, B and C lie on the circumference of a circle with centre O. DA is the tangent to the circle at A. BCD is a straight line. OC and AB intersect at E.

Angle BOC = 80°. Angle CAD = 38°.

a) Calculate the size of angle BAC.

b) Calculate the size of angle OBA.

c) Give a reason why it is not possible to draw a circle with diameter ED through the point A. [Edexcel]
10. A, B, C and D are points on the circumference of a circle centre O. A tangent is drawn from E to touch the circle at C. Angle AEC = 36°. EAO is a straight line.

a) Calculate the size of angle ABC. Give reasons for your answer. 
b) Calculate the size of angle ADC. Give reasons for your answer. [Edexcel]

11. P, Q and R are points on a circle. O is the centre of the circle. RT is the tangent to the circle at R. Angle QRT = 56°.

a) Find (i) the size of angle RPQ and (ii) the size of angle ROQ.

A, B, C and D are points on a circle. AC is a diameter of the circle. Angle CAD = 25° and angle BCD = 132°.

b) Calculate (i) the size of angle BAC and (ii) the size of angle ABD. [Edexcel]
## KEY POINTS

### Basic circle properties

- A line segment drawn from the centre of a circle to the midpoint of a chord will intersect the chord at right angles.
- A tangent and radius meet at right angles.
- The two external tangents to a circle are equal in length.

### Circle theorems

- The angle subtended by an arc at the centre of a circle is twice the angle subtended by the same arc at the circumference of the circle.
- Angles subtended by an arc in the same segment of a circle are equal.
- The angle subtended in a semicircle is a right angle.
- The angles subtended in opposite segments add up to 180°.
- The angle between a tangent and chord is equal to the angle subtended in the opposite segment.
Internet Challenge 19

The nine-point circle theorem

This diagram shows the nine-point circle. Here are the instructions to make it.

Start with any general triangle, whose vertices are A, B and C.

Construct points P1, P2, P3. Can you see what rule is used to locate them?

Construct the point M. Can you see how P1, P2, P3 are used to do this?

Construct points P4, P5, P6. Can you see what rule is used to locate them?

Construct points P7, P8, P9. Can you see what rule is used to locate them?

Then it should be possible to draw a circle that passes through all nine of the points: P1, P2, P3, P4, P5, P6, P7, P8 and P9.

Look at the diagram, and see if you can figure out how the various points are constructed. Use the internet to check that your deductions are correct.

Try to make a nine-point circle of your own, using compass constructions. You might also try to do this using computer graphics software.

Which mathematician is thought to have first made a nine-point circle?

Can you find a proof that these nine points all lie on the same circle?